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Properties of Soluble Subgroups of General Linear Group

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Abstract: In this paper we will determined the Properties of Soluble Subgroups of General Linear Group.

Keywords: Symplectic Groups, General Linear Groups, Primitive Groups.

INTERODUCTION AND ELEMANTRY DEFINITIONS

Dickson (1901, Chapter 12, pp. 260-287) determined all subgroups of PSp (2, pk) and in (1904) he determined all subgroups of PSp (4, 3). Mitchell (1914) determined the maximal subgroups of PSp(4,pk) for odd p. Liskovec (1973) classified the maximal Irreducible (p,q)- subgroups of GL (r2,p), where q and or are primes and q is odd. Colon (1977) determined the non - abelian q - subgroups (q prime) of GL (q,pk) and the non - abelion 2 - subgroups of Sp (2,pk). Harada and yamaki (1979) determined the irreducible subgroup of GL (n, 2) for n À 6. KondratÈev (1985, 1986, and 1987) determined the irreducible subgroups of GL (7, 2), the insoluble irreducible subgroups of GL (8,2) and GL (9,2) and the insoluble primitive subgroups of GL(10,2). In the early 1960 Sims developed an algorithm, based on coset enumeration, which takes as input a group G given by a finite representation and positive integer n, and output a list containing a representative of each conjugacy class of subgroups of G whose index is at most n. A similar algorithm was developed independently by Schaps (1968). After Kovacs, Neubuser and Newman (unpublished notes) have proposed an algorithm which computing certain maximal subgroups of low index. Now in this paper we will determine the irreducible Soluble Subgroups of Gl(4,pk). For this purpose, we mention some Definitions and elemantry notions.1.1 definition: let G,N and H be groups and G has a normal subgroup N0 isomorphic to N such that G/N0 is isomorphic to H, then we write G=N H. If G has a subgroup isomorphic to H which intersects N0 trivially then G is a semidirect product of N and H, we denote by G=N*H.

Definition

We say that 1.2 Definition:We say that a group G has a central decomposition (H1,...,Hn) if
1 - Each Hi is a normal subgroup of G
2 - G = H1 ... Hn.
3 - For each i and j, Hi Hj □ Z (Hi) Z (Hi)
4 - For each i and j, Hi Hj = Z (Hi) or Z (Hj)

We also say that G is the central product of Hi by G = H1Y ... YHn.

Definition

The holomorph of group G, denote by HOL (G), is the semidirect product of G and its automorphism group.

NOTATIONS AND ELEMENTARY RESULTS

In this Section we discuss some necessary results, which needed for later sections to Notations and elementary notions for use of their in after chapter.

Notation

We use sym(X) by means the symetric group on the set X, and Sn to means that the symetric group on the set of the first n positive integers. If G and H are permatotion groups, we denote the wreath product of G and H by G wr H, where G is a co-ordinate subgroup and H is the top group.

Theorem

(Huppert (1967, Theorem II. 3. 2. p. 159) Let P be a prime, n top a positive integer and V be the vector space of dimention n over the field of p elements. If G is

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a subgroup of GL (V), denote by V * G the permntation group of degree pn which is the semidirect product of V (acting on itself by translation) and G (acting in natural way) considered as asubgroup of sym(V). Let S be a complete and irredundant set of conjugacy class representatives of the irreducible soluble subgroups of GL (V).
(a) If G S. Then V * G is a primitive soluble permutation group of degree pn.
(b) If G S and H is a subgroup of GL (V), that is conjugate to G, then V * H is conjugate in sym (V) to V * G.
(c) If G S and H is a subgroup of Gl(V), that is not conjugate to G, then V * H is not conjugate to V * G.
(d) If P is a primitive soluble subgroup of sym(V), then there is a group G in S such that V * G is cojugate to P.

We always take F to be a finite field with pk elements and n a positive integer.

Theorem (See [4], [5] & [6] (: There exists an irreducible cyclic subgroup of order m in GL(n,F) if and only if m divides pkn -1 and m dos not devide pkd -1, for any positive integer d<n.

Theorem (See [4], [5] & [6]): If there exist irreducible cyclic subgroups of order m in GL(n,F) then they lie in a single conjugacy class.

Definition

In GL(n,F) the irreducible cyclic subgroups of order Pkn -1 are called the signer cycles.

Definition

An extraspecial q-group is a finite non abelian q-group whose centre, derived group and Frattini subgroup coincide and have order q.

Theorem: (See [3])

Let G be an ectraspecial q - group of order q1+2L and exponent q or 4. The group of automorphism S of G which acts trivially on both Z(G) and G/Z(G) is equal to Inn(G). Let H be the normal subgroup of Aut(G) consisting of those elements that act trivially on Z(G).

Then H/Inn(G) is isomorphic to a subgroup of the symplectic group Sp (2l,q).

If q is odd, then H/Inn(G) is isomorphic to the full symplectic group sp (2l, q), If G is the central product of l copies of D8, then H/Inn(G) is isomorphic to the orthogonal group O+(2l,2). If G is the central product of (l.1) copies of D8 and one Q8, then H/Inn(G) is isomorphic to the orthogonal group O-(2l, Z)

Let G be the central product of a cyclic group of order 4 and extraspecial 2-group. The group of antomorphisms of G that act trivially on both Z(G) andGZ(G) is equal to Inn(G). If H is the normal subgroup of Aut(G) consisting the those elements that act trivially on Z(G), then H/Inn(G) is isomorphic to the symplectic group Sp (2l,2).

Note that the group O+(2l,2) is the group of all linear transformations that preserve the quadratic form: f(x1, ... , x2l) = x1x2 + ... + x2l-1 x2l and the group O-(2l,2) is the group of all linear transformations that preserve the quadratic form: f(x1, ... , x2l) = x1x2 + ... + x2l-1 x2l + x22l-1 + x22l.

Definition: Let G be an irreducible subgroup of GL(n, F), acting on the vector space V. We call G imprimitiv if there exists a decomposition V=V1+ ...+Vr(r>1) of V that is preserred under the action of G.

We call the set {V1 ... V2} a system of imprimitivity for G, and each member of this set is called a block of imprimivity for G. The minimum of the set of dimensions of the blocks of imprimitivity for G is called the minimal block size of G. If G is not imprimitive. We call G primitive.

Theorem E :(see suprunenko (1976, theorem 15. 4. P.109) [11])

let M be an Imprimitive Maximal soluble subgroup of GL(n,F), and let □:= {V1...V2} be an unrefinable system of imprimitivity for M. Let : M □ Sym(□) be the homomorphism defined by: □g M, Vi (g):=Vig. Then NM(V1)|V1 is a primitive maximal soluble subgroup of GL(V1), M is a transitive maximal soluble subgroup of sym(□), and M is linearly isomorhic to NM (V1)|V1 Wr M .

Remark: Consider the case when m = n in the above theorem, then V1 is 1 -dimensional, and so NM (V1)|V1 = GL(l,F), by hypothesis M is irreducible, and therefore we must have pk > 2. In particular GL(n, 2) contains no imprimitive subgroups if n is prime.

Theorem (see [4]): Let m be aproper divisor of n, let Pm be a completete and irredundant set of conjugacy class representatives of the primitive maximal soluble subgroups of GL(m, F) and let be a complete and irredundant set of congugacy class represtatives of the transitive maximal soluble subgroups of . Define the set Sm of imprimitive soluble subgroups of GL(n,F) by Sm:={P wr T|P□Pm, T□ }.

However, if pk = 2, then define the set S1 to be empty. Let S be the union of the Sm as m runs through the proper divisors of n. Then those members of S that are maximal soluble from a complete and irredundant set of conjugacy class representatives of the imprimitive maximal soluble subgroups of Gl (n, F).

Theorem: (See [13], lemma 19. 1. P.129, Theorem 20. 9, P.145). Let A be a maximal abelian normal subgroup of M. Then the following statements hold:

- A is conjugate to a group of block diagonal matrices, where each blak is the same, and is m by m, where m is a divisor of n:
- The linear span E, of the powers of any one of the m by m diagonal blocks of A is an extension field of FIm.
- The degree of this field extension is m.

- A is isomorphic to the multiplicative group of E: in particular, A is cyclic of order pkm -1.
- A is the unique maximal abelian normal subgroup of M.

Notation: Define the map □: NL (F)□ GL(2l,q) by

Where: Theorem (See to [4], [5] & [6]):

Let n = ql m. Where l>0, and q is a prime divisor of pkm-1. If q = 2, then suppos in addition that Pkm □1(mod4). Let z1 be our fixed generator of a singer cycle of GL (m, F), and let a1 be our fixed element of order m in GL (m, F) such that . Let a, z be the n by n block diagonal matrices with a1 and z1 runing down their diagonals, respectively. Define the matrices ui and vi as Notatin 2.13 . Let S be the subgroup of GL (2l, q) That is generated by Sp (2l,q) and the block diagonal matrix with the matrix running down its diagonal. Let D be a completely reducible (not necessarily maximal) soluble subgroup of S which does not fix any non - zero isotropic subspace of the natural module for Sp (2l,q).

Suppose D has generating set {d1,...,dr}. If di is the matrix then gi be any matrix of GL(n, F) statistying for some (arbitrary) integer □j and □j. let P the subgroup of GL(n, F) defined by: P:=<G<a>(v1,...,vl),g1,...,gr,u1.v1,...,ul,vl,z> then P is the complete inverse image of D under r. Furthermore, P is primitive and has a maximal abelian normal subgroup of order Pkm-1. Now let D1 be a complete and irredundant set of S – conjugacy class representatives of the completely reducible maximal soluble subgroup of S which do not fix any non - zero isotropic subspace of the natural module for sp (2l,q). Let P1 be the set of groups P obtained by the above method, one for each D, where D runs through the members of D1. No two members of P are conjugate in GL(n, F). If M is a primitive maximal soluble subgroup of GL(n, F) whose unique maximal abelian normal subgroup has order Pkm-1, then M is conjugate to a member of P1.

Theorem: Let n = 2lm, and suppose that pkm □3(mod4). Let z be our fixed generator of a singer cycle of GL(m, F), and let a1 be our fined element of order m in GL(m,F) such that . Let a and z be the n by n block diagonal matriees with a1 and z1 running down their diagonals, respectively For 1□i□l-1 define the matrices ui and vi as notation 2.13.

Define ul+ and vl+ by: and and definee. ul- and vl- by and Where □ and □ are two elements of FIm such that □2+□2= -Im. Let D be a completely reducible (not necessarily maximal) soluble subgroup of O+(2l,2) or O-(2l,2) which does not fix any non-zero isotropic subspace of the natural module for the relevant orthogonal group. Suppose D has generating set {d1,....,dr}. If di is the matrix Then let gi be any matrix of GL(n,F) statistying for some (arbitrary) integers □j and □j, and where the superscript * is replaced by + or - according as D belongs to O+(2l,2) or O-(2l,2), respectively, let P be the subgroup of GL(n,F) defined by P:=<C<a> (v1,...,vl*), g1,...,gr,u1,v1,...,u1*,vl*,z> then P is the complete inverse image of D under r. Furthermore, P is primitive and has a maximal abelian normal subgroup of order Pkm-1. now let D+ be a complete and irredundant set of O+(2l,2)- conjugacy class representatives of the completely reducible maximal soluble subgroups of O+(2l,2) which do not fix any non-zero isotropic subspace of the natural module for O+(2l,2).

Define D- similarly, with O-(2l,2) in place of O+(2l,2), let P1 be the set of groups P obtained by the above method, one for each D, where D runs through the members of D+ and D-. No two members of P1 are conjugate in GL(n,F). If M is a primitive maximal soluble subgroup of GL(n,F) whose unique maximal a belian normal subgroup hos order Pkm-1. Then M is conjugate to a member of P1.

Proof: The proof of this theorem goes exactly the similar with the proof previous theorem and the reader can be referred to [4] & [5].

Definition: Any group constructed by the methods theorems 2.11, 2.14 and 2.15 will be called a JS-maximal (for jordan-suprunenko) of GL(n,F). we will also use the terms JS-imprimitive and JS-primitive to denote imprimitive and primitive JS-maximal, respectively. Note that every JS-maximal is irreducible and soluble. but not necessarily maximal soluble. The smallest value of pkn for which there are JS-maximals and are not maximal soluble is 9. (see [11]).

Remark: For the imprimitives we use P wr T where P and T are as described in Theorem A and For the primitives, we use (Y E)N D where E is extraspeical of order q1+2l and exponent q or 4, and D is as described in theorem B or C. Of course, there may be many (pairwise non - isomorphic) groups with than a normal subgroup isomorphic to Y E whose quotient is isomorphic to D. But we always mean the one obtained by the construction methods in this paper.

Definition: Let the JS - maximal S of GL(n,F) be M1,...,Mm and let G be an irreducible soluble subgroup of GL(n, F). If n is prime and G is cyclic, then we define the guardian of G to be that JS - maximal which is the normaliser of a singer cycle. Otherwise the gunrdian of G is defined to be Mi, where i is the least positive interger such that G is GL(n,F) - conjugate to a subgrup of M.

In this Section by using previous theorems and methods of Section 1 and section 2, We determine The irreducible Soluble Subgroups of GL(4,pk). For this we proof the following main theorem.

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Theorem: Let p be a prime number and k,n be positive integers and let F be the field of pk elements. Then the number of types of The irreducible Soluble Subgroups in the group GL(4,pk) is 10.

Proof: By definition 2.16 since every JS-maximal is irreducible and Soluble, therefore, we suffices to determine the JS-maximal soluble subgroups of the GL(4,pk). For this purpose, let F be the field of pk elements. Since both GL(1,F) and S2 are Soluble, therefore by theorem (2.11) and remark (2.17) there is exactly one Js-imperative soluble subgroup of GL(2,F), namely, M1(2,pk):=GL(1,pk) wr S2 , pk□2 Also since the unique maximal abelian normal subgroup of the group GL(2,pk) order p2k-1 or pk-1, then by theorems 2.14, 2.15 and remark 2.17 the JS-Primitive group of order p2k-1 and pk-1,as follows: M2(2,pk): = * C2, M3(2,pk): = (Y Q8) N O-(2,2), pk □3(mod4) M4(2,pk): = (Y Q8) N Sp(2,2), pk □1(mod4)

Therefore by used from JS - maximals GL(2,pk), the JS-imprimitives of GL(4,pk) listed as follows.
M1(4,pk): = GL(1, pk) wr S4,
M2(4,pk): = M2(2,pk) wr S2,
M3(4,pk): = M3(2,pk) wr S2 , pk □3(mod4),
M4(4,pk): = M4(2,pk) wr S2 , pk □1(mod4),

And also The JS - primitive of Gl(4,pk) are listed below:
M5(4,pk): = * C4,
M6(4,pk): = M2(2,pk) * C2, pk □2,
M7(4,pk): = Y D8 Y Q8)N O+(4,2) , pk □ 3(mod4),
M8(4,pk): = Y D8 Y Q8)N HOL(C5) , pk □3(mod4),
M9(4,pk): = Y D8 Y Q8)N O+(4,2) , pk □ 1(mod4),
M10(4,pk): = Y D8 Y Q8)N HOL(C5) , pk □1(mod4).

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